S-2338

M.A./M.Sc. (IVth Semester)

Examination, 2022-23

MATHEMATICS

[Paper - II]

[Functional Analysis]

Time : 2½ Hours] **[Maximum Marks : 80**

Note : This question paper consists of two sections, Section A and B. Attempt any four questions each from section 'A' and 'B'. Limit your answers within the given answer book. B answer book will not be provided.

SECTION—A

(Short Answer Type Questions) **4×5 = 20**

1. Prove that on a finite a finite dimensional space norms are equivalent.

2. Define Cauchy sequence and show that in a normed linear space, every convergent sequence is a Cauchy sequence.

3. Prove that a non empty subset S of a normed linear space N is bounded if and only if f(S) is a bounded set of numbers for each $f \in N^*$.

4. Let X be a Banach space and let Y be a normed space over the field K. If a set F of bounded linear operator from X to Y is pointwise bounded, then prove that it is uniformly bounded.

5. If $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is any vector in H, then show that the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

6. If x and y are any two vectors in a Hilbert space H, then prove that :

   $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

7. Prove that the adjoint operator $T \to T^*$ on B(H) has the following properties :

   (a) $(T_1 + T_2)^* = T_1^* + T_2^*$

   (b) $(T_1 T_2)^* = T_2^* T_1^*$

8. An operator T on a Hilbert space H is normal iff :

   $\|T^* x\| = \|Tx\|$

SECTION—B

(Long Answer Type Questions) **4×15 = 60**

1. (a) Show that the linear space e^∞ equipped with the normal given by :

   $\|x\|_\infty = \sup_{1 \leq i < \infty} |\xi_i|$, $x = \{\xi_i\} \in e^\infty$ is a Banach space.

(b) Let N be a normed linear space and $x, y \in N$ then :

$\|x\| - \|y\| \leq \|x - y\|$

2. (a) Let N be a non-zero normed linear space and $S = \{x \in N : \|x\| = 1\}$ be a linear subspace of N. Prove that N is Banach space if S is complete.

(b) Let N be a normed linear space and M a subspace of N. Then show that the closure $\overline{M}$ of M is also a subspace of N.

3. (a) Let M be a closed linear subspace of a normed linear space and x_0 a vector not in M. Then prove that there exists functional F in N^* such that :

$F(M) = \{0\}$ and $F(x_0) \neq 0$

(b) Let B and B′ be Banach spaces. If $T : B \rightarrow B'$ in continuous linear transformation of B onto B′, then prove that T is open mapping.

4. (a) Let B and B′ be Banach spaces and let T be linar transformation of B into B′. Then prove that T is continuous mapping if and only if its graph is closed.

(b) Let X and Y be normed spaces over the field K and $T : X \rightarrow Y$ a bounded (continuous) linear operator. Then show that the null space N(T) is closed.

5. (a) If T is an operator on a Hilbert space H, then prove that :

$\langle T_x, x \rangle 0$ for all x with if and only if $T = 0$

(b) If M is a closed linear subspace of a Hilbert space H, then prove that :

$H = M \oplus M^{\perp}$

6. (a) Let S be a nonempty subset of Hilbert space H. Then prove that orthogonal complement $S^{\perp}$ is a closed linear subspace of H.

(b) Let L be an inner product space. Show that $\sqrt{(x, x)}$ has the properties of a norm.

7. (a) If P is a projection on a Hilbert space H with range M and null space N, then prove that $M \perp N$ if and only if P is self-adjoint.

(b) If T is a normal operator on a finite dimensional Hilbert space H, then prove that there exists an orthonormal basis for H relative to which the matrin of T is diagonal.

8. (a) Show that an arbitrary operator T on a Hilbert space H can be uniquely expressed as $T = T_1 + i T_2$ Where T_1 and T_2 are self-adjoint operators on H.

(b) If T is an arbitrary operator on a Hilbert space H, and if α and β are scalars such that :

$|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal.

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